Denotational Semantics

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Lectures for Part II CST 2026/2027

Recap of Lecture 3

- We defined posets, cpos, and domains.
- We defined monotone and (strict) continuous functions.

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- We defined monotone and (strict) continuous functions.
- We proved Kleene's fixed point theorem:
  Every continuous function on a domain has a least (pre)fixed point.
- We saw methods for constructing domains:
  - flat domains $X_\bot$ on a set ($\mathbb{B}_\bot$ and $\mathbb{N}_\bot$)
  - products of domains $\prod_{i \in I} D_i$
  - separated sums $\sum_{i \in I} D_i$

Constructions on Domains

Function domains

Cpo/domain of continuous functions

Given two cpos (domains) $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$, the function cpo $(D \to E, \sqsubseteq)$ has underlying set

$$\{f : D \to E \mid f \text{ is a } \textit{continuous} \text{ function}\}$$

equipped with the pointwise order:

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Argument-wise least elements and lubs:

$$\bot_{D \to E}(d) = \bot_E \qquad \left(\bigsqcup_{n \in \mathbb{N}} f_n\right)(d) = \bigsqcup_{n \in \mathbb{N}} f_n(d)$$

Function operations are continuous

The following operations on continuous functions are well-defined and continuous:

- Evaluation

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- Currying of a continuous $f : D' \times D \to E$:

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- Composition

$$\begin{aligned} \circ &: (E \to F) \times (D \to E) \to (D \to F) \\ f \circ g &= \lambda d \in D.\ g(f(d)) \end{aligned}$$

Continuity of the fixed point operator

Proposition

The least fixed point operator $\mathrm{fix} : (D \to D) \to D$ is continuous.

- We need to show that fix is monotone:
  If $f \sqsubseteq g$, then $f(\mathrm{fix}\, g) \sqsubseteq g(\mathrm{fix}\, g) \sqsubseteq \mathrm{fix}\, g$, so $\mathrm{fix}\, f \sqsubseteq \mathrm{fix}\, g$
- We need to show it preserves lubs of chains, that is, $\mathrm{fix}(\bigsqcup_n f_n) \sqsubseteq \bigsqcup_n \mathrm{fix}(f_n)$. This follows by the RHS being a fixed point of $\bigsqcup_n f_n$:

$$\left(\bigsqcup_n f_n\right)\left(\bigsqcup_k \mathrm{fix}(f_k)\right) = \bigsqcup_n \bigsqcup_k f_n(\mathrm{fix}(f_k)) = \bigsqcup_k f_k(\mathrm{fix}(f_k)) = \bigsqcup_k \mathrm{fix}(f_k)$$

Constructions on Domains

Back to the introduction

The semantics of a while loop

$$[\![\texttt{while } X > 0 \texttt{ do } (Y := X * Y; X := X - 1)]\!]$$

is a fixed point of the following $F : D \to D$, where D is $(\mathrm{State} \rightharpoonup \mathrm{State})$:

$$F(w)([X \mapsto x, Y \mapsto y]) \quad = \quad \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ w([X \mapsto x - 1, Y \mapsto x \cdot y]) & \text{if } x > 0 \end{cases}$$

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$$F(\perp) = \perp$$

This is continuous and $\text{State}_\perp \to \text{State}_\perp$ is a domain!

Kleene's fixed point theorem

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We can compute explicitly

$$w_\infty[X \mapsto x, Y \mapsto y] = \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ [X \mapsto 0, Y \mapsto (x!) \cdot y] & \text{if } x > 0 \end{cases}$$

We can check this agrees with the operational semantics.

Scott Induction

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Let D be a domain, $f : D \to D$ be continuous, and $S \subseteq D$. If the set S

(i) contains $\bot$,

(ii) is chain-closed, *i.e.* the lub of any chain of elements of S is also in S,

(iii) is stable for f, *i.e.* $f(S) \subseteq S$,

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$$\textsc{ScottInd}\ \frac{\Phi(\bot) \qquad \Phi(x) \Rightarrow \Phi(f(x)) \qquad (\forall i \in \mathbb{N}.\, \Phi(x_i)) \Rightarrow \Phi(\bigsqcup_{i \in \mathbb{N}} x_i)}{\Phi(\mathrm{fix}(f))}$$

Building chain-closed sets

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In other words, any formula built using $\vee, \wedge, \forall, \sqsubseteq, =$ and continuous f defines chain-closed subsets.

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Any formula written using:

- signature: continuous functions + constants
- relations: equality, inequality
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Given any set I, domains D, E, functions $(f_i)_{i\in I}$, $g\colon D \to E$, $e \in E$,

$$\Phi(x) \coloneqq \forall y \in E, (\forall i \in I, f_i(x) \sqsubseteq y) \vee g(x) = e$$

is chain-closed.

Example: downset

Proposition

Assume $f(d) \sqsubseteq d$, *i.e.* d is a pre-fixed point of a continuous $f : D \to D$. Then $\mathrm{fix}(f) \sqsubseteq d$.

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Proof by Scott induction on $d\downarrow$

Example: partial correctness

Let $w_\infty : \mathrm{State}_\perp \to \mathrm{State}_\perp$ be the denotation of

$$\texttt{while } X > 0 \texttt{ do } (Y := X * Y; X := X - 1)$$

Recall that $w_\infty = \mathrm{fix}(F)$ where

$$F(w)(x, y) = \begin{cases} (x, y) & \text{if } x \leq 0 \\ w(x - 1, x \cdot y) & \text{if } x > 0 \end{cases}$$
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Claim:

$$\forall x.\ \forall y \geq 0.\ w_\infty(x, y) \Downarrow \implies \pi_Y\left(w_\infty(x, y)\right) \geq 0$$

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Proof: by Scott induction!